Optimized Components in Frame Synthesis

LEWIS P. FELTON* AND RICHARD B. NELSON†
University of California, Los Angeles, Calif.

The incorporation of optimized thin-walled beam elements into procedures for the minimum-weight design of indeterminate elastic planar frame structures is examined. It is shown that all frame structures composed of beam elements with thin-walled cross sections will be fully stressed in the sense that each element should be proportioned to be on the verge of local instability under at least one independent load condition. This fact leads to a formulation of the design requirements as an inequality-constrained minimization problem with the only variables being moments of inertia of the elements. Solutions are examined numerically by use of a nonlinear programing technique. The formulation also suggests a simple and efficient direct iterative procedure for obtaining designs which are fully-stressed with respect to both local buckling and yielding. For the examples considered, both procedures are shown to produce identical results.

Nomenclature

A = cross-sectional area
D = depth of beam cross section
E = Young's modulus
I = moment of inertia
K = buckling coefficient
M = bending moment
$\mathfrak{M}$ = maximum absolute value of M in element
W = weight
Z = section modulus
k_A,k_I = constants
l = length
t = thickness
ρ = specific weight
σ_Y = yield stress
$\bar{\sigma}$ = $\mathfrak{M}/Z$

Introduction

IN the field of optimum structural design two basic approaches have been found to have the most widespread applicability. The first of these approaches is concerned with the special techniques pertaining to the minimum-weight design of thin-walled structural components subject to failure by instability.[1] Optimization is achieved by proportioning cross-sectional dimensions for simultaneous occurrence of all considered failure modes at maximum load.

The second approach utilizes the techniques of mathematical programing for the study of large-scale structural system designs which have been formulated as inequality-constrained optimization problems. These synthesis techniques have been used in truss, plate, shell and frame designs.[2–5]

The latter approach has demonstrated that, for certain design problems, a minimum-weight structure need not be "fully-stressed"; i.e., need not have each element develop its maximum allowable stress under at least one of several independent load conditions.

The occurrence of nonfully stressed minimum-weight structures designed using the programing techniques, and the inherently fully-stressed nature of components optimized by the direct techniques noted above, seems to have produced a tendency to regard the two approaches as incompatible. In apparently only one reported study[6] has it been shown that "preoptimized" thin-walled components can be advantageously incorporated into structural synthesis procedures, specifically in conjunction with truss design. Interaction between the two approaches has not been previously investigated in conjunction with frame design. In fact, previous studies of frame synthesis have generally not dealt with thin-walled sections subject to instability.

It is the purpose of this paper to illustrate the use of preoptimized thin-walled components in the minimum-weight design of a broad class of elastic planar frame structures under single and multiple load conditions. The components considered are assumed to be prismatic thin-walled cross-sectional members which are adequately characterized by the Bernoulli-Euler beam theory, which neglects the effects of axial deformations. It is also assumed in this study that each cross section is symmetric about its neutral axis and that the member material is ideally elastic-plastic with the same absolute value of the yield stress in both tension and compression. Under these conditions it will be shown that in any frame with elements possessing given stiffnesses each member must be proportioned to be on the verge of local instability (buckling or yielding) under at least one of the multiple load conditions. This fact, used in conjunction with a simplifying approximation, allows formulation of the design problem as an inequality-constrained minimization problem for which the only variables are the moments of inertia of each member rather than a set of linear cross-sectional dimensions for each member. This problem is then examined numerically by use of a nonlinear programming technique.

Alternately, the fully-stressed nature of the optimized thin-walled beams suggests a very simple and efficient iterative procedure for obtaining fully-stressed frames, every element of which is on the verge of local instability at yield stress.

The results of the two design methods are compared and shown to lead to identical designs for all examples considered; i.e., frames in which every member is on the verge of local instability at the yield stress under at least one load condition.

Optimization of Thin-Walled Beams

Before considering frames in detail, a discussion of minimum-weight designs for thin-walled beams is presented. For any thin-walled beam, the cross section of which is symmetric about its neutral axis and is completely defined by specification of two variables, t, a thickness, and D, a depth, where $D \gg t$, the section properties may be given by

$$A = k_A D t \tag{1a}$$

Received May 18, 1970; revision received September 3, 1970.
The authors wish to thank L. D. Hofmeister and M. W. Dobbs for providing portions of the computer programs used in this study.

* Assistant Professor, Mechanics and Structures Department. Member AIAA.
† Assistant Professor, Mechanics and Structures Department. Associate Member AIAA.

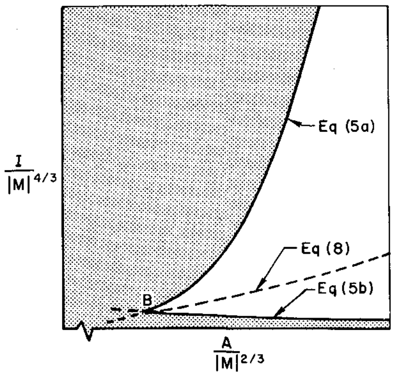

Fig. 1 Design space for thin-walled beam.

$$Z = 2k_I D^2 t \tag{1b}$$

$$I = k_I D^3 t \tag{1c}$$

where A, Z, and I denote cross-sectional area, section modulus, and moment of inertia, respectively, and k_A, and k_I are constants.

The beam material is assumed ideally elastic-plastic with a modulus of elasticity E and yield limit σ_Y in both tension and compression. For acceptable cross-sectional designs the maximum absolute value of bending stress, $\bar{\sigma}$, must be limited by the relations

$$\bar{\sigma} \leqq KE(t/D)^m = KE(k_I/k_A{}^2)^m(A^2/I)^m \tag{2a}$$

$$\bar{\sigma} \leqq \sigma_Y \tag{2b}$$

where K and m are empirical or theoretical constants specified for each cross-sectional shape. The quantity $KE(t/D)^m$ represents the critical bending stress at which elastic instability occurs. Relation (2b) applies for elastic designs for which

$$KE(t/D)^m > \sigma_Y \tag{3}$$

and expresses beam instability at yield. Relations (2) express only local cross-sectional instabilities and are independent of beam length.

It is then possible to generate the optimum cross-sectional area A, given the magnitude of the specified moment, $|M|$, since

$$\bar{\sigma} = |M|/Z \equiv (k_A{}^{1/2}/2k_I{}^{1/2})|M|/(I^{1/2}A^{1/2}) = (k_A{}^{1/2}/2k_I{}^{1/2})(|M|^{4/3}/I)^{1/2}(|M|^{2/3}/A)^{1/2} \tag{4}$$

Substitution of Eq. (4) in relations (2) gives

$$I/|M|^{4/3} \leqq (2KEk_I{}^{(2m+1)/2}/k_A{}^{(4m+1)/2})^{2/(2m-1)} \times (A/|M|^{2/3})^{(4m+1)/(2m-1)} \tag{5a}$$

and

$$I/|M|^{4/3} \geqq k_A/4k_I\sigma_Y{}^2(A/|M|^{2/3})^{-1} \tag{5b}$$

Relations (5) are presented in terms of special forms of the "structural index" for a beam[1]; i.e., $I/|M|^{4/3}$ and $A/|M|^{2/3}$. The significance of the structural index lies in the fact that all geometrically similar beam cross sections having identical structural indices will develop identical stress distributions. Consequently, the critical bending stress given by Eq. (2a), which is a function of the dimensionless ratio t/D, will be the same for all sections as will the optimum stress which is a particular value of critical stress. Thus, relations (5) are applicable to any family of geometrically similar beams, the cross-sectional shape and behavior of which is defined by particular values of the constants.

The form of a graphical solution to relations (5) is shown schematically in Fig. 1. The curves representing the equalities, Eqs. (5a) and (5b), and the unshaded region in Fig. 1, constitute acceptable solutions to relations (5). Point B in Fig. 1 represents the minimum-weight design for any cross section at which bending moment is specified.

The significance of this approach and the resulting insight it provides with regard to frame design will be examined in the next section.

Optimization of Indeterminate Frames

Consider the minimum-weight design of a statically indeterminate frame structure composed of n thin-walled beam elements of the type just presented. The layout of the structure is assumed specified, that is, the length l_i of each member and its location in space is given, as well as its cross-sectional form and material properties. The frame then has a total of $2n$ design variables, values D_i, and t_i for each of the n members. (Alternate sets I_i and A_i; or I_i and t_i/D_i, etc., may also be used.)

The structure is assumed loaded by q independent load conditions so that in each member i there are q different maximum bending moments $|M_{ij}|$, where $i = 1,\ldots,n$ is the member number and $j = 1,\ldots,q$ is the load condition number. Let

$$\mathfrak{M}_i = \max_j |M_{ij}|, \qquad i = 1,\ldots,n, \qquad j = 1,\ldots,q \tag{6a}$$

be the largest absolute value of moment in the ith member under any of the q load conditions. The maximum absolute value of stress is then

$$\bar{\sigma}_i = \mathfrak{M}_i/Z_i, \qquad i = 1,\ldots,n \tag{6b}$$

For the ith member of a general indeterminate frame, the value $\mathfrak{M}_i$ is a function of I_k, $k = 1,\ldots,n$. However, relations (5a) and (5b) still apply for each component provided that the quantity $|M|$ which appears is replaced by $\mathfrak{M}_i$, and Fig. 1 remains valid. For this reason significant conclusions can be drawn regarding the behavior of the minimum-weight frame.

Suppose that at some stage in the design process a complete design, which may not be optimum, is obtained. This implies that all A_i and I_i are specified, and that corresponding to the distribution I_k is a distribution of bending moment. Thus, $\mathfrak{M}_i$ is known for each element, and the design parameters for the element are then representable by a point in the space shown in Fig. 1, the coordinates of which are $I_i/\mathfrak{M}_i{}^{4/3}$ and $A_i/\mathfrak{M}_i{}^{2/3}$. Assume that the particular design of element i, i.e., (A_i, I_i) is not constrained by either Eq. (5a) or (5b), but rather lies within the acceptable (unshaded) region. It follows that A_i may be reduced, without altering I_i, until element i is on the verge of local instability below or at the yield stress. This corresponds to moving the design of the element along a horizontal line from right to left in Fig. 1. Since moment of inertia remains constant, frame stiffness and bending moment distribution also remain constant.

It is concluded that a minimum-weight frame will be obtained when all thin-walled beam elements which comprise the frame have their cross-sectional areas reduced until they are on the verge of instability under at least one critical load condition. Thus, any frame design problem may now be stated

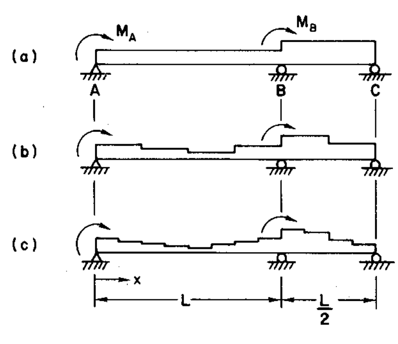

Fig. 2 Two-span beam examples [in (b) segment length = $L/4$; in (c) segment length = $L/8$].

in the form

$$\text{minimize:} \quad W = \rho \sum_{i=1}^{n} A_i l_i \tag{7a}$$

$$\text{subject to:} \quad \bar{\sigma}_i = KE(t_i/D_i)^m \text{ for } D_i/t_i \geqq (KE/\sigma_Y)^{1/m}, \quad i = 1,\ldots,n \tag{7b}$$

$$\bar{\sigma}_i = \sigma_Y \text{ for } D_i/t_i \leqq (KE/\sigma_Y)^{1/m}, \quad i = 1,\ldots,n \tag{7c}$$

where ρ = specific weight and W = total structural weight. For convenience, the quantities K, E, σ_Y, m, and ρ have been assumed constant for all members although the developments can be generalized to accommodate different values for each member.

For the present study only the set of fully-stressed designs for which

$$D_i/t_i \geqq (KE/\sigma_Y)^{1/m}, \qquad i = 1,\ldots,n \tag{8}$$

will be considered. This statement is equivalent to replacing the yield constraint, Eq. (5b), by the parabola through point B shown in Fig. 1. However, as a consequence of the previously noted fact that all designs may be moved horizontally from right to left in Fig. 1, the effect of Eq. (8) is actually to remove from the design space only those designs with values of I below a horizontal line through point B. The use of relation (8) leads to a particularly simple problem formulation, and restricts the design space only slightly, especially for problems involving thin-walled members made of materials with high-yield stresses.

Under restriction (8), relations (7b) and (7c) become

$$\bar{\sigma}_i \equiv \mathfrak{M}_i/Z_i = KE(t_i/D_i)^m, \qquad i = 1,\ldots,n \tag{9a}$$

and

$$KE(t_i/D_i)^m \leqq \sigma_Y, \qquad i = 1,\ldots,n \tag{9b}$$

Noting the expressions

$$Z_i = 2k_I^{1/4}I_i^{3/4}(t_i/D_i)^{1/4} \tag{10a}$$

$$A_i = k_A k_I^{-1/2}(t_i/D_i)^{1/2}I_i^{1/2} \tag{10b}$$

Equation (9a) may be used to give t_i/D_i in terms of I_k, $k = 1,\ldots,n$,

$$(t_i/D_i) = (\mathfrak{M}_i/2KEk_I^{1/4}I_i^{3/4})^{4/(4m+1)} \tag{11}$$

Equation (11) may then be substituted into inequality (9b) and, by use of Eq. (10b), into Eq. (7a) giving the minimization problem in only the n variables I_k, $k = 1,\ldots,n$,

$$\text{minimize:} \quad W = \rho k_A k_I^{-1/2} \sum_{i=1}^{n} I_i^{1/2} \times (\mathfrak{M}_i/2KEk_I^{1/4}I_i^{3/4})^{2/(4m+1)} l_i \tag{12a}$$

$$\text{subject to:} \quad (\mathfrak{M}_i/2KEk_I^{1/4}I_i^{3/4})^{4m/(4m+1)} \leqq \sigma_Y/KE \qquad i = 1,\ldots,n \tag{12b}$$

Relations (12) may be solved numerically by means of a nonlinear programming search procedure. Such solutions are discussed in the next section.

Alternately, the preceding formulations suggest a simple direct iterative procedure which generates designs in which every frame member is stressed to the yield limit σ_Y under at least one load condition. Fully-stressed designs of this type are obtained by finding a set of I_k, $k = 1,\ldots,n$, such that

$$I_i/\mathfrak{M}_i^{4/3} = (KE)^{1/3m}/2^{4/3}k_I^{1/3}\sigma_Y^{(4m+1)/3m}, \qquad i = 1,\ldots,n \tag{13}$$

which is a form indicated by removing the ratio $A_i/\mathfrak{M}_i^{2/3}$ from relations (5) taken as equalities. The form of Eq. (13) is exactly the same as that obtained by considering relations (12b) as equalities and solving for the quantity $I_i/\mathfrak{M}_i^{4/3}$.

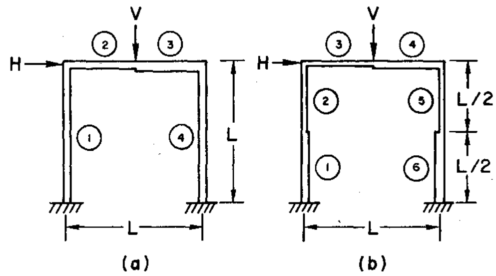

Fig. 3 Rigid frame examples.

The iterative procedure consists of modifying the design obtained at any step p, $p = 1,\ldots,N$, of the design sequence according to the relation

$$(I_i)_{p+1} = [(KE)^{1/3m}/2^{4/3}k_I^{1/3}\sigma_Y^{(4m+1)/3m}](\mathfrak{M}_i)_p^{4/3}, \qquad i = 1,\ldots,n \tag{14}$$

The resulting structure is one for which every member lies at point B of its design space (Fig. 1).

Examples

Several examples, involving single and multiple load conditions, were examined using both a mathematical programming solution technique and the iterative solution technique. Two basic structural configurations, as shown in Figs. 2 and 3, were considered.

Figure 2 shows a two-span continuous beam subjected to moments applied at the supports. Figure 3 shows a single-story, single-bay rigid frame subjected to concentrated forces. Two distinct loading cases were considered for each structure. In the first case the two forces were assumed to be applied simultaneously, and in the second case the forces were assumed to be applied independently. As indicated in Figs. 2 and 3, each structure was also segmented to varying degrees, and the resulting minimum-weight designs were computed.

All examples, except the beam shown in Fig. 2c which contains twelve design variables (moments of inertia of the twelve segments), were examined by numerically solving relations (12). Results were obtained by using an interior penalty function algorithm suggested by Fiacco and McCormick,[7] which transforms relations (12) into an unconstrained minimization problem. The penalty function has the form

$$P(X,r_p) = W(X) + r_p \sum_{i=1}^{2n} \frac{1}{g_i(X)} \tag{15}$$

where $X^T = \{I_1 I_2,\ldots,I_n\}$ is the vector of design variables, $W(X)$ is weight given by Eq. (12a), and r_p is a scalar. The first n of the $g_i(X)$ are the constraints given by relations (12b) rewritten in the form

$$g_i(X) \equiv 1.0 - \frac{KE}{\sigma_Y}\left(\frac{\mathfrak{M}_i}{2KEk_I^{1/4}I_i^{3/4}}\right)^{4m/(4m+1)} \geqq 0, \qquad i = 1,\ldots,n \tag{16a}$$

while constraints $n + 1,\ldots,2n$ are side constraints of the form

$$g_k(X) \equiv I_{k-n} \geqq 0, \qquad k = n + 1,\ldots,2n \tag{16b}$$

which are included to assure that all design variables will be positive.

Functions of the form given by Eq. (15) are solved for a sequence of values of r_p, $p = 1,\ldots,N$, and the resulting sequence of solutions, X_p converges to the solution of the original constrained problem. Each unconstrained problem of the sequence was minimized by the variable metric method.[8,9] The initial value r_1 was computed using criterion 1 of Ref. 7, and subsequent values of r_p were calculated from

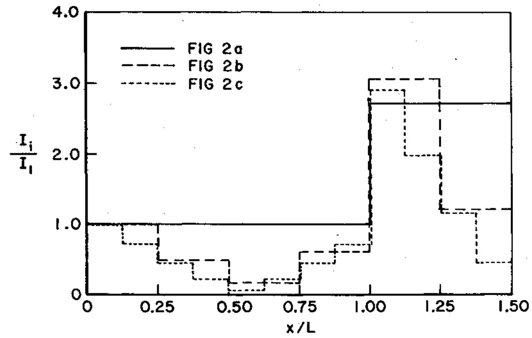

Fig. 4 Normalized moments of inertia for minimum-weight designs of beams shown in Fig. 2 ($M_B = 3M_A$, single load condition).

$r_{p+1} = r_p/c$, where c is a constant greater than 1.0. Gradients were evaluated using a standard forward-difference operator. This optimization procedure has been successfully used in previous studies[10] and further details will not be included here. Designs obtained from the solution of relations (12) were compared to those generated by the iterative procedure, Eq. (14). Sample calculations, which illustrate the simplicity and rapid convergence of the procedure, are included in the Appendix.

All designs (except those involving Fig. 2c) are presented in Tables 1 and 2 which indicate that, for the examples considered, both procedures give nearly identical results; i.e., all minimum-weight designs have each member constrained by local buckling and yield stress.

Other designs, fully stressed at yield, were also located by both the iterative and search methods, but these were local minima and are therefore not presented. It is interesting to note that the final results depend significantly upon the degree of segmentation specified, as shown by the symmetric four member frame design and the unsymmetric six member design obtained under identical cross-sectional, material, and loading assumptions.

The structural analysis routines used in the design procedures were based on a standard stiffness formulation in which constant axial stiffnesses were assigned to the frame members. As is indicated by the tables, the minimum-weight designs which result may have members of zero bending stiffness but finite axial stiffness.

The results presented in Tables 1 and 2 are based on assigned values of the loads, dimension, and constants $L = 40$ in., $M_A = 15$ in. $-k$, $M_B = 45$ in. $-k$, $H = 500$ lb, $V = 750$ lb, $E = 30 \times 10^6$ psi, $\sigma_Y = 30$ ksi, and $\rho = 0.3$ lb/in.^{3}. The cross section constants $K = 1.5$, $k_A = 3.0$, $k_I = 7/12$ and $m = 2.0$ apply for a square I-beam with equal depth and flange width and equal web and flange thicknesses.[1] However, results are applicable to all geometrically similar structural configurations subjected to the same types and ratios of forces, regardless of cross-sectional configuration; i.e., the relative stiffnesses of the elements of the optimized structure will remain the same for all thin-walled elements.

Figure 4 illustrates this point for the beams of Fig. 2. Design variables I_i have been normalized by dividing by I_1, which for this problem may be evaluated from Eq. (13) by setting $\mathfrak{M}_i = |M_A|$. Included in Fig. 4 are the results for the 12-segment beam shown in Fig. 2c. These results were readily obtained by the iterative procedure, and were not verified by the search procedure because of the excessive computational time which would have been required.

In general, it was found that, on an IBM 360/91, the time required to solve Eqs. (12) was on the order of 10 sec to 2 min, depending on whether the problem involved two or six variables. Convergence was slow in the vicinity of the optimum, and relatively large changes in variables could be obtained with little change in weight. By comparison, the twelve-variable problem was completed in less than 1 sec by the iterative method, thus indicating the superior efficiency of this method.

Conclusions

The combination of component and system optimization procedures has been shown to be advantageous in the study of the minimum weight design of a broad class of indeterminate frame structures. The simple iterative procedure has provided, for all examples considered, designs identical with those obtained by use of the programing method. However, in the absence of conclusive evidence regarding the exact nature of optimum structures of the type considered here, the results of the iterative method must be considered approximate subject to further research.

Appendix: Iterative Design Procedure

In order to illustrate the ease of applicability and rapid convergence of the iterative fully-stressed design procedure, computations for one example are presented below. Example: beam of Fig. 2a under simultaneous application of moments M_A and $M_B = 3M_A$. Let $\beta = I_2/I_1$. Since the problem involves only two design variables, it may conveniently be solved in terms of β.

The internal moments must be evaluated at the left and right ends of members AB and BC. These moments are denoted, respectively, by M_1, M_2, M_3, M_4 and their values are given by

$$|M_1| = |M_A| \quad \text{(A1a)}$$

Table 1 Design results; single load condition

Example	Solution method	Member number 1	2	3	4	5	6	W, lb
Fig. 2a[a]	Iteration: I_i, in.^{4}	0.638	1.725	...	...	...	...	10.985
	Search: I_i, in.^{4}	0.646	1.744	...	...	...	...	...
	g_i	0.002	0.002	...	...	...	...	11.065
Fig. 2b[a]	Iteration: I_i, in.^{4}	0.638	0.306	0.111	0.392	1.944	0.771	8.651
	Search: I_i, in.^{4}	0.639	0.307	0.110	0.387	1.952	0.776	8.684
	g_i	0.000	0.001	0.000	0.000	0.000	0.000	...
Fig. 3a[b]	Iteration: I_i, in.^{4}	0.000	0.064	0.355	0.389	...	...	7.914
	Search: I_i, in.^{4}	0.000	0.064	0.356	0.390	...	...	7.965
	g_i	0.492	0.001	0.001	0.001	...	...	...
Fig. 3b[b]	Iteration: I_i, in.^{4}	0.000	0.000	0.000	0.638	0.638	0.147	7.476
	Search: I_i, in.^{4}	0.000	0.000	0.000	0.638	0.638	0.148	7.514
	g_i	0.485	0.369	0.224	0.000	0.000	0.000	...

[a] Segment numbering sequence starts from left end of beam (see Fig. 2).
[b] See Fig. 3 for segment numbering sequence.

Table 2 Design results[a]; multiple load conditions

Example	Solution method	Member number 1	2	3	4	5	6	W, lb
Fig. 2a[b]	Iteration: I_i, in.^{4}	0.638	2.324	...	...	...	...	11.782
	Search: I_i, in.^{4}	0.645	2.346	...	...	...	...	11.861
	g_i	0.002	0.001	...	...	...	...	...
Fig. 2b[b]	Iteration: I_i, in.^{4}	0.638	0.361	0.131	0.183	2.620	1.040	9.108
	Search: I_i, in.^{4}	0.638	0.361	0.131	0.182	2.623	1.042	9.141
	g_i	0.000	0.000	0.000	0.000	0.000	0.000	...
Fig. 3a[c]	Iteration: I_i, in.^{4}	0.184	0.139	0.139	0.184	...	...	9.289
	Search: I_i, in.^{4}	0.184	0.139	0.139	0.184	...	...	9.322
	g_i	0.000	0.000	0.000	0.000	...	...	...
Fig. 3b[c]	Iteration: I_i, in.^{4}	0.635	0.149	0.220	0.220	0.000	0.000	8.003
	Search: I_i, in.^{4}	0.646	0.155	0.227	0.224	0.002	0.000	8.297
	g_i	0.005	0.007	0.007	0.004	0.183	0.100	...

[a] Values of g_i are given for active constraints only.
[b] Segment numbering sequence starts from left end of beam (see Fig. 2).
[c] See Fig. 3 for segment numbering sequence.

$$|M_2| = \left|2M_A\left[1 + \frac{1}{2}\left(\frac{1-4\beta}{1+2\beta}\right)\right] + \frac{M_A}{2}\left(1 + \frac{0.5}{\beta}\right)^{-1}\right| \quad \text{(A1b)}$$

$$|M_3| = \left|M_A\left[1 - \left(\frac{1-4\beta}{1+2\beta}\right)\right] - \frac{M_A}{2}\left(1 + \frac{0.5}{\beta}\right)^{-1}\right| \quad \text{(A1c)}$$

$$|M_4| = 0 \quad \text{(A1d)}$$

As an initial design assume $I_2 = I_1$; i.e., $\beta_1 = 1.0$.
Step 1. From Eqs. (A1a) to (A1d):

$$|M_1| = |M_A| \quad \text{(A2a)}$$

$$|M_2| = 1.333|M_A| \quad \text{(A2b)}$$

$$|M_3| = 1.667|M_A| \quad \text{(A2c)}$$

$$|M_4| = 0 \quad \text{(A2d)}$$

Therefore, the maximum moments in members 1 and 2 are given by

$$\mathfrak{M}_i = |M_2| = 1.333|M_A| \quad \text{(A3a)}$$

$$\mathfrak{M}_2 = |M_3| = 1.667|M_A| \quad \text{(A3b)}$$

From Eq. (14)

$$(I_1)_{p+1} = \Lambda(\mathfrak{M}_1)_p{}^{4/3} \quad \text{(A4a)}$$

$$(I_2)_{p+1} = \Lambda(\mathfrak{M}_2)_p{}^{4/3} \quad \text{(A4b)}$$

where Λ represents the constant in brackets on the right side of Eq. (14). It follows from Eqs. (A4a) and (A4b) that

$$\beta_{p+1} = (\mathfrak{M}_2/\mathfrak{M}_1)_p{}^{4/3} \quad \text{(A5)}$$

or

$$\beta_2 = (1.667/1.333)^{4/3} = 1.347 \quad \text{(A6)}$$

The successive values of β are:

Step 2 $\beta_3 = (1.825/1.175)^{4/3} = 1.799$ (A7)

Step 3 $\beta_4 = (1.956/1.044)^{4/3} = 2.31$ (A8)

Step 4 $\beta_5 = (2.057/1.000)^{4/3} = 2.61$ (A9)

Step 5 $\beta_6 = (2.093/1.000)^{4/3} = 2.68$ (A10)

Step 6 $\beta_7 = (2.106/1.000)^{4/3} = 2.70$ (A11)

The value of β_7 is accurate to three significant figures. Given β, numerical values of I_1, I_2 are computed by use of Eqs. (A4).

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